THE ACTION OF S_n ON THE COHOMOLOGY OF $\overline{M_{0,n}}(\mathbb{R})$

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Abstract. In recent work by Etingof, Henriques, Kamnitzer, and the author, a presentation and explicit basis was given for the rational cohomology of the real locus $\overline{M_{0,n}}(\mathbb{R})$ of the moduli space of stable genus 0 curves with n marked points. We determine the graded character of the action of S_n on this space (induced by permutations of the marked points), both in the form of a plethystic formula for the cycle index, and as an explicit product formula for the value of the character on a given cycle type.

1. Introduction

For any integer $n \geq 3$, let $\overline{M_{0,n}}$ be the moduli space of stable curves of genus 0 with n marked points; by convention, for $n = 1$, $n = 2$, this is just a single point, but we never allow $n = 0$. Since a stable curve of genus 0 has trivial automorphism group, this is in fact a smooth projective scheme over $\mathbb{Z}$ (and a *fine* moduli space), and thus its real locus $M_n := \overline{M_{0,n}}(\mathbb{R})$ is a smooth compact manifold. The symmetric group acts on M_n by permuting the marked points, and thus acts on the cohomology. The main result of the present work is an explicit product formula for the (graded) character of this action.

Theorem 1.1. *Let $\pi \in S_n$ be a permutation with $n_1 + 1$ fixed points and n_m m-cycles for $m > 1$, and define*

$$o_m = \sum_{1\leq k:2^k|m} (2^{-k}m) n_{2^{-k}m}.$$

Then

$$\sum_k (-t)^k \operatorname{Tr}(\pi|H^k(M_n,\mathbb{Q})) = \prod_{1\leq l}(\gamma_l(t) + o_l t^{l/2}) \prod_{0\leq i\leq n_l-2} (\gamma_l(t) + (o_l + l(n_l - 2 - 2i))t^{l/2}),$$

where the polynomials $\gamma_l(t)$ satisfy

$$\sum_{\text{odd } k|l} t^{-l/2k}\gamma_{l/k}(t) = t^{-l/2}.$$

Remarks. 1. Note that we are using the standard convention for products with negatively many terms; thus for $n_l \le 0$,

$$\prod_{0\le i\le n_l-2} (\gamma_l(t) + (o_l + l(n_l - 2 - 2i))t^{l/2}) := \prod_{n_l-1\le i\le -1} (\gamma_l(t) + (o_l + l(n_l - 2 - 2i))t^{l/2})^{-1}.$$

In particular, the infinite product is indeed well-defined, since if $n_l = 0$, the corresponding factor is 1. For $n_1 = -1$, the corresponding factor is

$$\prod_{-2\le i\le -1} (1 + (-3 - 2i)t^{1/2})^{-1} = 1/(1 - t).$$

Similarly, the presence of $t^{l/2}$ for l odd is not an issue, since then the corresponding factor is invariant under $t^{l/2} \to -t^{l/2}$ (simply reverse the order of multiplication in the product over i, and note that $o_l = 0$). Finally, $\gamma_l(t)$ is indeed a polynomial, since by Möbius inversion,

$$t^{-l/2}\gamma_l(t) = \sum_{\text{odd } k|l} \mu(k)t^{-l/2k},$$

and thus

$$\gamma_l(t) = \sum_{\text{odd } k|l} \mu(k)t^{l(1-1/k)/2}.$$

2. Also note the factor $(-1)^k$ above; in particular, the Euler character of M_n is given by setting $t = 1$ above (or taking a limit, if $n_1 = -1$). In this context, it is worth noting that $\gamma_l(1) = 0$ unless l is a power of 2, and $\gamma_{2^k}(t) = 1$.

3. On the identity element, we obtain

$$\prod_{0\le i\le n-3} (1 + (n - 3 - 2i)t^{1/2}) = \prod_{0\le i\le \lfloor (n-3)/2\rfloor} (1 - (n - 3 - 2i)^2 t),$$

agreeing with the formula of [4] for the Poincaré series of M_n.

4. We finally note that the above formula is remarkably similar to the following formula of Lehrer [10, 9], valid for $n \ge 3$:

$$\sum_k (-t)^k \operatorname{Tr}(\pi|H^k(M_{0,n}(\mathbb{C}), \mathbb{Q})) = (1 - t)^{-1} \prod_{1\le l} \prod_{0\le i<n_l} (\eta_l(t) - lit^l),$$

where the polynomials $\eta_l(t)$ satisfy

$$\sum_{k|l} t^{-l/k}\eta_l(t) = t^{-l}.$$

Of course, the close analogy between the cohomology of these spaces was already noted in [4].

As one might imagine from the form of the above result, it is much more natural to consider the action of S_{n-1} on M_n, rather than the full action of S_n. Indeed, the results of [4] on the structure of $H^*(M_n, \mathbb{Q})$ (summarized in Section 2) give a particularly nice description of this restriction in terms of the homology (not *co*homology, as one would normally expect) of a certain poset; the corresponding character was studied in [2]. In Section 3, by combining these results, we obtain an expression (Theorem 3.5) for the "cycle index" of the restriction, i.e., a generating function for the character. In Section 4, we derive a number of differential equations satisfied by the cycle index; the corresponding recurrences for the character prove the theorem for the restriction (i.e., when π has a fixed point). Finally, in Section 5, we show that $H^*(M_n, \mathbb{Q})$ satisfies a particularly strong form of functoriality which in particular enables us to derive the full S_n character from the S_{n-1} character alone, proving the main theorem. (We also give an expression for the corresponding cycle index (Theorem 5.4).) Finally, in Corollary 5.5, we give a formula for the Euler character of M_n, in particular determining the precise permutations for which the Euler character is nonzero.

Notation

As we are dealing with cohomology, it will be convenient to use "super" conventions. That is, if $V_1, \dots V_n$ is a sequence of graded vector spaces (with it being understood here and in the sequel that the coefficient field is $\mathbb{Q}$ and all nontrivial homogeneous components have finite dimension and nonnegative degree), we identify the two tensor products

$$V_1 \otimes V_2 \otimes \cdots \otimes V_n$$

and

$$V_{\pi(1)} \otimes V_{\pi(2)} \otimes \cdots \otimes V_{\pi(n)}$$

for any permutation π via the isomorphism

$$v_1 \otimes v_2 \otimes \cdots \otimes v_n \to \prod_{i<j, \pi(i)>\pi(j)} (-1)^{\deg(v_i)\deg(v_j)} v_{\pi(1)} \otimes v_{\pi(2)} \otimes \cdots \otimes v_{\pi(n)}$$

for any sequence of homogeneous elements $v_i \in V_i$. Similarly, if A is a graded algebra, we say that it is supercommutative if

$$xy = (-1)^{\deg(x)\deg(y)} yx.$$

In particular, the free supercommutative algebra generated by elements of degree 1 is simply the exterior algebra.

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2. The cohomology of M_n

Theorem 2.1. [4] *For $n \geq 1$, the algebra $\Lambda_n := H^*(M_n, \mathbb{Q})$ is the supercommutative quadratic algebra generated over $\mathbb{Q}$ by elements $\omega_{ijkl}, 1 \leq i, j, k, l \leq n$, antisymmetric in $ijkl$, with defining relations*

$$\omega_{ijkl} + \omega_{jklm} + \omega_{klmi} + \omega_{lmij} + \omega_{mijk} = 0$$

and

$$\omega_{ijkl}\omega_{ijkm}$$

for any distinct i, j, k, l, m. Moreover, the action of S_n on $H^*(M_n, \mathbb{Q})$ is given in terms of these generators by

$$\pi^*(\omega_{ijkl}) = \omega_{\pi(i)\pi(j)\pi(k)\pi(l)}.$$

This extends naturally to a functor $\Lambda : \mathbf{Bij}^+ \to \mathbb{Q}\text{-}\mathbf{GrAlg}$, where $\mathbf{Bij}^+$ is the category of nonempty finite sets and bijections, and $\mathbb{Q}$-**GrAlg** is the category of graded $\mathbb{Q}$-algebras. As we mentioned in the introduction, we will need to consider also a restriction of this to the category **Bij** of all finite sets and bijections.

Proposition 2.2. [4] *For any ordered finite set S, let $\Lambda'(S)$ denote the supercommutative algebra generated by antisymmetric elements ν_{ijk} for distinct $i, j, k \in S$ subject to the relations*

$$\nu_{ijk}\nu_{ijl} = 0$$

and

$$\nu_{ijk}\nu_{klm} + \nu_{jkl}\nu_{lmi} + \nu_{klm}\nu_{mij} + \nu_{lmi}\nu_{ijk} + \nu_{mij}\nu_{jkl} = 0;$$

extend this to a functor $\mathbf{Bij} \to \mathbb{Q}\text{-}\mathbf{GrAlg}$ by

$$\Lambda'(\pi)(\nu_{ijk}) = \nu_{\pi(i)\pi(j)\pi(k)}.$$

Then for each $n \geq 0$, there is an isomorphism $\Lambda'(\{1, 2, \dots, n\}) \cong \Lambda_{n+1}$ defined on generators by

$$\nu_{ijk} \mapsto \omega_{ijkn}.$$

A monomial in the generators ν_{ijk} determines an equivalence relation on S (taking $i \cong j \cong k$ if ν_{ijk} appears in the monomial); equivalently, each monomial determines a partition of S into (unordered) disjoint subsets. If ρ is such a partition (a fact denoted by the relation $\rho \vdash S$), let $\Lambda'[\rho]$ denote the span in $\Lambda'(S)$ of all monomials corresponding to ρ; note that $\Lambda'[\rho]$ is unchanged (up to canonical isomorphism) if we remove a singleton class from ρ and S. In particular, we may let $\Lambda'[T]$ denote the case in which ρ has a single nontrivial equivalence class, equal to T; the result is independent of S up to canonical isomorphism.

Theorem 2.3. [4] *The spaces $\Lambda'[\rho]$ for different ρ are linearly independent, and thus*

$$\Lambda'(S) = \bigoplus_{\rho \vdash S} \Lambda'[\rho].$$

If ρ has classes ρ_1, ρ_2,..., ρ_k, then multiplication in $\Lambda'(S)$ induces a natural isomorphism

$$\Lambda'[\rho_1, \rho_2, \ldots, \rho_k] \cong \Lambda'[\rho_1] \otimes \Lambda'[\rho_2] \otimes \cdots \otimes \Lambda'[\rho_k];$$

this remains valid even if some singleton classes of ρ are omitted.

Finally, the indecomposable spaces $\Lambda'[T]$ can be expressed in terms of certain poset homology groups.

Theorem 2.4. [4] *If $|T|$ is even, then $\Lambda'[T] = 0$; otherwise, if $|T| = 2n+1$,*

$$\Lambda'[T] \cong \tilde{H}_n(\Pi_T^{\mathrm{odd}}, \mathbb{Q}) \otimes \mathrm{sgn},$$

where Π_T^{odd} is the poset of partitions of T with all parts odd, $\tilde{H}_n$ is the top (shifted) reduced homology of this poset, and sgn *is the sign representation of* $\mathrm{Sym}(T)$.

Remark. Note that in $\tilde{H}_n$, the degree has been shifted by 1 from the standard definition of poset homology, in order to obtain the correct degree in $\Lambda'[T]$. In any event, $\tilde{H}_n$ is the only nontrivial homology group of Π_T^{odd} (which is Cohen-Macaulay [1, 2]), so there is no risk of confusion.

3. Cycle indices

Let $\mathbb{Q}$-**GrVect** denote the category of graded vector spaces W and degree 0 linear transformations. Given an endomorphism $\phi : W \to W$ in $\mathbb{Q}$-**GrVect**, the graded trace of ϕ is defined to be the power series $\mathrm{Tr}(\phi) \in \mathbb{Q}[[t]]$ defined by

$$\mathrm{Tr}(\phi)(t) := \sum_{k \geq 0} t^k (-1)^k \mathrm{Tr}_{W_k}(\phi);$$

the sign factor reflects our interpretation of W as a graded superspace.

Now, let V be a representation of **Bij** in $\mathbb{Q}$-**GrVect** (a "graded representation of **Bij**").

Definition. The *cycle index* of V is the power series $Z_V \in \mathbb{Q}[[t, p_1, p_2, \ldots]]$ given by

$$\sum_{n \geq 0} \frac{1}{n!} \sum_{\pi \in S_n} \mathrm{Tr}(V(\pi))(t) \prod_i p_i^{n_i(\pi)}$$

where for a permutation π, $n_i(\pi)$ is the number of i-cycles of π.

Remark. We may similarly associate a cycle index to an arbitrary virtual (graded) character of **Bij** (i.e., a sequence χ_n such that χ_n is a virtual character of S_n).

There are two natural gradings on the above algebra of power series (t-degree and p-degree), defined on generators by

$$\deg_t(t) = 1, \deg_t(p_i) = 0, \deg_p(t) = 0, \deg_p(p_i) = i;$$

a cycle index Z_V is homogeneous of t-degree d if V is homogeneous of degree d, and homogeneous of p-degree d if $V(S) = 0$ for $|S| \neq d$.

The sum and product of cycle indices is itself a cycle index, as is

$$F^{\sim} := F(t, p_1, -p_2, p_3, -p_4, \dots).$$

Proposition 3.1. *Let V and W be two graded representations of* **Bij**. *Then*

$$Z_V + Z_W = Z_{V \oplus W}, \qquad Z_V Z_W = Z_{V \cdot W}, \qquad Z_V^{\sim} = Z_{V \otimes \mathrm{sgn}}$$

where

$$(V \oplus W)(S) = V(S) \oplus W(S),$$
$$(V \cdot W)(S) = \bigoplus_{T \subset S} V(T) \otimes W(S \setminus T),$$

extended to functors in the natural way.

There is a further operation known as plethysm (or composition), which on two series F and G with $G(t, 0, 0, \dots) = 0$ is defined as

$$F[G] := F(t, G(t, p_1, p_2, \dots), G(t^2, p_2, p_4, \dots), \dots);$$

this is easily verified to be an associative (but not commutative or distributive) operation. We will also need the obvious extension of this to series involving fractional powers of t.

Proposition 3.2. *For any graded representations V and W of* **Bij** *such that $W(\emptyset) = 0$, we have*

$$Z_V[Z_W] = Z_{V[W]},$$

where $V[W]$ is the graded representation with

$$V[W](S) := \bigoplus_{\rho \vdash S} V(\rho) \otimes \bigotimes_i W(\rho_i),$$

extended in the natural way to a functor.

Remark. If W is supported on sets of a given cardinality, this is essentially classical; for the general case, see for instance [8, Thm. 6.5].

In general, plethysm does not interact well with tensoring with the sign character; there is, however, one important special case.

Proposition 3.3. *If every term of the series G has odd p-degree, then*

$$F^{\sim}[G^{\sim}] = F[G]^{\sim}.$$

There are three particularly important cycle indices. For the trivial representation, we have

$$\mathrm{Exp} := Z_{\mathrm{triv}} = \exp\Big(\sum_{i \geq 1} p_i / i\Big).$$

In particular, $\mathrm{Exp}[Z_V]$ is the cycle index of the functor

$$S \mapsto \bigoplus_{\rho \vdash S} \bigotimes_i V(\rho_i).$$

We will also need analogues of the hyperbolic sine and cosine:

$$\begin{aligned}\mathrm{Cosh} &:= \frac{\exp(\sum_{i\geq 1} p_i/i) + \exp(\sum_{i\geq 1}(-1)^i p_i/i)}{2}\\ \mathrm{Sinh} &:= \frac{\exp(\sum_{i\geq 1} p_i/i) - \exp(\sum_{i\geq 1}(-1)^i p_i/i)}{2}.\end{aligned}$$

The corresponding representations are obtained from the trivial representation by removing the spaces associated to sets with odd or even cardinality, respectively.

We can now state Calderbank, Hanlon, and Robinson's result on the homology of Π_n^{odd}.

Theorem 3.4. [2] *The cycle index of the functor* $\tilde{H}_*(\Pi_T^{\mathrm{odd}}, \mathbb{Q})$ *is*

$$(1 - \mathrm{Cosh}[\mathrm{Arcsinh}[t^{1/2}p_1]]) + (t^{-1/2}\,\mathrm{Arcsinh}[t^{1/2}p_1]),$$

where Arcsinh *is the unique symmetric function such that*

$$\mathrm{Sinh}[\mathrm{Arcsinh}] = \mathrm{Arcsinh}[\mathrm{Sinh}] = p_1.$$

Note that the first term gives the cycle index for $|T|$ even, while the second term gives the cycle index for $|T|$ odd. Also, since Sinh is concentrated in odd p-degree, the same is true of Arcsinh, and thus

$$\mathrm{Sinh}^{\sim}[\mathrm{Arcsinh}^{\sim}] = \mathrm{Arcsinh}^{\sim}[\mathrm{Sinh}^{\sim}] = p_1^{\sim} = p_1.$$

This then gives us our first result on the action of S_n on $H^*(M_n, \mathbb{Q})$.

Theorem 3.5. *The cycle index of the functor* Λ' *is*

$$\mathrm{Exp}[t^{-1/2}\,\mathrm{Arcsinh}^{\sim}[t^{1/2}p_1]]$$

where $\mathrm{Arcsinh}^{\sim}$ *is the unique symmetric function such that*

$$\mathrm{Sinh}^{\sim}[\mathrm{Arcsinh}^{\sim}] = \mathrm{Arcsinh}^{\sim}[\mathrm{Sinh}^{\sim}] = p_1.$$

Proof. Since

$$\Lambda'(S) \cong \bigoplus_{\rho\vdash S} \Lambda'[\rho] \cong \bigoplus_{\rho\vdash S}\bigotimes_i \Lambda'[\rho_i] \cong \bigoplus_{\substack{\rho\vdash S\\ \text{all parts odd}}} \bigotimes_i (\tilde{H}_n(\Pi_S^{\mathrm{odd}}, \mathbb{Q}) \otimes \mathrm{sgn}),$$

it follows that $Z_{\Lambda'} = \mathrm{Exp}[Z_V]$, where for $|S|$ odd,

$$V(S) = \tilde{H}_n(\Pi_S^{\mathrm{odd}}, \mathbb{Q}) \otimes \mathrm{sgn} = \tilde{H}_*(\Pi_S^{\mathrm{odd}}, \mathbb{Q}) \otimes \mathrm{sgn}\,.$$

But tensoring with sgn simply applies the homomorphism $p_i \to (-1)^{i-1}p_i$ to the cycle index; the result follows. □

It seems appropriate to mention in passing the corresponding formula for the cohomology of the *complex* moduli space.

Theorem 3.6. [7, 8] *The cycle index of the functor* $S \mapsto H^*(\overline{M_{0,|S|+1}}(\mathbb{C}), \mathbb{Q})$ *is given by* $\mathrm{Exp}[C]$ *where* C *is the plethystic inverse of*

$$\frac{t^{-2}(\mathrm{Exp}[t^2p_1] - 1) - t^2(\mathrm{Exp} - 1)}{1 - t^2}.$$

Proof. The argument of Theorem 4.5 of [4] for computing the Poincaré series from (essentially) the S_{n-1}-invariant basis of [13] extends immediately to the level of cycle indices. We thus find that the cycle index is of the form $\mathrm{Exp}[C]$ where C satisfies

$$C = p_1 + \sum_{m\geq 3} \sum_{1\leq l\leq m-2} t^{2l} h_m[C];$$

here h_m is the cycle index of the trivial representation of S_m, or equivalently the p-degree m component of Exp. Moving the sum to the left-hand side, we find that this indeed specified C as a plethystic inverse; simplifying the geometric sum gives the desired result. □

Remarks. 1. It does not appear to be feasible to obtain an explicit formula for the graded character (unlike the real case, as we will shortly see); indeed, it appears that no formula is known for the Poincaré series, let alone any other values of the character.

2. In the references, this is expressed as 1 plus the plethystic inverse of

$$\mathrm{Exp}[t^2 \,\mathrm{Log}[1+p_1]]/(t^4-t^2) - 1/(t^4-t^2) - p_1/(t^2-1),$$

where $\mathrm{Log}[1+p_1]$ denotes the plethystic inverse of $\mathrm{Exp}-1$. This in turn is essentially the cycle index of the cohomology of the *un*-compactified moduli space $M_{0,|S|+1}(\mathbb{C})$, suggesting that there should be a cohomological interpretation for the plethystic inverse

$$t^{-1/2}\,\mathrm{Sinh}^{\sim}[t^{1/2}\,\mathrm{Log}[1+p_1]]$$

of the cycle index for $H^*(M_{|S|+1})$.

4. The explicit graded character

It turns out that by using some ideas from [2], we can actually obtain an explicit formula for the graded character, rather than a mere generating function. It will be convenient to introduct another symmetric function

$$X := \sum_{k \text{ odd}} \mathrm{Arcsinh}^{\sim}[p_k]/k.$$

It follows from the definition of Arcsinh that X is the plethystic inverse of the series

$$\frac{\exp(p_1 - \sum_{k>0} p_{2^k}/2^k) - \exp(-p_1 - \sum_{k>0} p_{2^k}/2^k)}{2},$$

and thus is a function of p_1, p_2, p_4,... alone (since that subalgebra is closed under plethysm). Similarly,

$$\begin{aligned} C :=&\ \mathrm{Cosh}^{\sim}[\mathrm{Arcsinh}^{\sim}] \\ =&\ \frac{\exp(X - \sum_{k>0} X[p_{2^k}]/2^k) + \exp(-X - \sum_{k>0} X[p_{2^k}]/2^k)}{2} \end{aligned}$$

also depends only on the variables p_{2^k}.

Lemma 4.1. *The function X satisfies the differential equation*

$$2^l C[p_{2^l}]\frac{\partial}{\partial p_{2^l}}X = \delta_{l0} + \sum_{0\le k<l} 2^k p_{2^k}\frac{\partial}{\partial p_{2^k}}X.$$

Proof. If we differentiate the plethystic equation

$$\frac{\exp(X - \sum_{k>0} X[p_{2^k}]/2^k) - \exp(-X - \sum_{k>0} X[p_{2^k}]/2^k)}{2},$$

we find

$$2^l C\frac{\partial}{\partial p_{2^l}}X = \delta_{l0} + p_1\sum_{0\le k<l}(2^k\frac{\partial}{\partial p_{2^k}}X)[p_{2^{l-k}}],$$

so in particular the claim holds for $l=0$. For $l>0$, if we multiply both sides by $C[p_{2^l}]/C$, we find by induction that

$$\begin{aligned}
2^l C[p_{2^l}]\frac{\partial}{\partial p_{2^l}}X &= \frac{p_1}{C}\sum_{0\le k<l}(2^k C[p_{2^k}]\frac{\partial}{\partial p_{2^k}}X)[p_{2^{l-k}}]\\
&= \frac{p_1}{C} + \sum_{0\le j<k<l}(2^j p_{2^j}\frac{\partial}{\partial p_{2^j}}X)[p_{2^{l-k}}]\\
&= \frac{p_1}{C} + \sum_{0\le j<k<l}(2^j p_{2^j}\frac{\partial}{\partial p_{2^j}}X)[p_{2^{k-j}}]\\
&= \frac{p_1}{C} + \sum_{0<k<l} p_{2^k}2^k\frac{\partial}{\partial p_{2^k}}X.
\end{aligned}$$

∎

Lemma 4.2. *Let c_1, c_2, c_3,… be indeterminates, and define a symmetric function*

$$G := \exp(\sum_{k\ge 0} c_k X[p_k]/k).$$

Let G_l be the result of setting $p_k = 0$ for $k > l$. Then G_l satisfies the differential equations

$$\Big(c_l + \sum_{\substack{1\le k\\ 2^k|l}} 2^{-k}lp_{2^{-k}l}\frac{\partial}{\partial p_{2^{-k}l}}\Big)^2 G_l = l^2\Big(p_{2l}\frac{\partial}{\partial p_l}\Big)^2 G_l + l^2\Big(\frac{\partial}{\partial p_l}\Big)^2 G_l$$

and

$$l\frac{\partial}{\partial p_l}G_l\,\Big|_{p_l=0} = \Big(c_l + \sum_{\substack{1\le k\\ 2^k|l}} 2^{-k}lp_{2^{-k}l}\frac{\partial}{\partial p_{2^{-k}l}}\Big)G_{l-1}.$$

Proof. From the previous lemma, linearity, and the fact that $X[p_k]$ depends only on the variables $p_{2^j k}$, we find that

$$lC[p_l]\frac{\partial}{\partial p_l}\log(G) = c_l + \sum_{\substack{1\le k\\ 2^k|l}} 2^{-k}lp_{2^{-k}l}\frac{\partial}{\partial p_{2^{-k}l}}\log(G),$$

and thus

$$lC[p_l]\frac{\partial}{\partial p_l}G = \Big(c_l + \sum_{\substack{1\le k\\ 2^k|l}} 2^{-k} l p_{2^{-k}l}\frac{\partial}{\partial p_{2^{-k}l}}\Big)G,$$

The second differential equation is immediate (since $C[0]=1$); for the first equation, we have (note that if we set $p_2 = p_3 = \cdots = 0$ in C, we obtain the function $\cosh(\operatorname{arcsinh}(p_1)) = \sqrt{1+p_1^2}$)

$$l\sqrt{1+p_l^2}\frac{\partial}{\partial p_l}G_l = \Big(c_l + \sum_{\substack{1\le k\\ 2^k|l}} 2^{-k} l p_{2^{-k}l}\frac{\partial}{\partial p_{2^{-k}l}}\Big)G_l.$$

Since the differential operators on either side commute with each other, we in fact have

$$\Big(l\sqrt{1+p_l^2}\frac{\partial}{\partial p_l}\Big)^2 G_l = \big(c_l + \sum_{\substack{1\le k\\ 2^k|l}} 2^{-k} l p_{2^{-k}l}\frac{\partial}{\partial p_{2^{-k}l}}\big)^2 G_l,$$

which simplifies to the desired equation. $\square$

Lemma 4.3. *Suppose χ is a virtual character of* **Bij** *with cycle index*

$$Z_\chi = \exp(\sum_{l\ge 1} c_l X[p_l])$$

for some sequence c_l independent of p_1, p_2,.... Let π be a permutation, and for each $m>0$ let $n_m(\pi)$ be the number of m-cycles of π; also define

$$o_m(\pi) = \sum_{\substack{1\le k\\ 2^k|m}} 2^{-k} m n_{2^{-k}m}(\pi).$$

Then

$$\chi = \prod_{l\ge 1}(c_l+o_l)\prod_{0\le i\le n_l-2}(c_l+o_l+l(n_l-2-2i)).$$

Proof. Note that if $n_l = 0$, the inner product is over -1 terms, and is thus by standard convention equal to $(c_l+o_l)^{-1}$, so the product over l is well-defined.

In terms of the cycle index, χ is given by

$$\chi(n_1,n_2,\dots) = \Big(\prod_{1\le l}(l\frac{\partial}{\partial p_l})^{n_l}\Big)Z_\chi\,\Big|_{p_1=p_2=\cdots=0},$$

where we view χ as a function of the values $n_i(\pi)$. In particular, the lemma can be interpreted as giving recurrences for the character; we find that, if $n_{m+1} = n_{m+2} = \cdots = 0$,

$$\chi(n_1,\dots,n_m+2,0,0,\dots) = ((c_m+o_m)^2 - (mn_m)^2)\chi(n_1,\dots,n_m,0,0,\dots)$$

and similarly from the second differential equation of the lemma,

$$\chi(n_1,\dots,n_{m-1},1,0,0,\dots) = (c_m+o_m)\chi(n_1,\dots,n_{m-1},0,0,0,\dots).$$

But then by induction on the sequence n_i, in reverse lexicographic order, the given character formula follows. $\square$

Remark. The special case $c_1 = -c_2 = -c_4 = -c_8 = \cdots = \lambda$, all other $c_i = 0$, was shown in [2, Thm. 5.7], via a rather different argument.

In particular, the cycle index of Λ' is of this form, and we thus obtain the following.

Theorem 4.4. *Let π be a permutation with $n_m(\pi) = n_m$ for $m \geq 1$. Then*

$$\begin{aligned}\chi_{\Lambda'}(\pi; t) :=& \operatorname{Tr}(\Lambda'(\pi)) \\ =& \prod_{1\leq l}(\gamma_l(t) + o_l t^{l/2}) \prod_{0\leq i\leq n_l-2}(\gamma_l(t) + (o_l + l(n_l - 2 - 2i))t^{l/2}),\end{aligned}$$

where the polynomials $\gamma_l(t)$ are given by the expression

$$\gamma_l(t) = \sum_{\substack{k|l \\ k \text{ odd}}} \mu(k) t^{l(1-1/k)/2},$$

where μ is the Möbius function.

Proof. This is equivalent to the claim

$$Z_{\Lambda'} = \exp\Big(\sum_{1\leq l} t^{-l/2}\gamma_l(t) X[t^{l/2}p_l]/l\Big)$$

since then we can apply the lemma to $Z_{\Lambda'}[t^{-1/2}p_1]$.

The claimed expression for $Z_{\Lambda'}$ is easily obtained by expanding

$$\begin{aligned}\sum_{1\leq l} t^{-l/2} \operatorname{Arcsinh}^{\sim}[t^{l/2}p_l]/l &= \sum_{1\leq l} t^{-l/2} \sum_{m \text{ odd}} \mu(m) X[t^{lm/2}p_{lm}]/lm \\ &= \sum_{1\leq l} \sum_{\substack{m|l \\ m \text{ odd}}} t^{-(l/m)/2}\mu(m) X[t^{l/2}p_l]/l.\end{aligned}$$

$\square$

Corollary 4.5. *Theorem 1.1 holds whenever $n_1 \geq 0$.*

Proof. If $n_1 \geq 0$, or in other words if π has a fixed point (so WLOG $\pi(n) = n$), then this follows immediately from the isomorphism between $\Lambda'(\{1, 2, \ldots, n-1\})$ and $H^*(M_n, \mathbb{Q})$. $\square$

5. Functoriality

In fact, as we will see, the character formula continues to hold even if π has no fixed point (so $n_1 = -1$). The key idea is that although we have so far only considered Λ as a functor on **Bij** (or more precisely on the category of *nonempty* finite sets and bijections), it actually extends to a functor on the full category $\mathbf{Fin}^+$ of nonempty sets.

For a nonempty finite set S, let $\Lambda(S)$ denote the algebra isomorphic to Λ_n with generators ω_{ijkl} for $i, j, k, l \in S$. This extends to a functor $\Lambda : \mathbf{Fin}^+ \to \mathbb{Q}\text{-}\mathbf{GrAlg}$ as follows. If $f : S \to T$ is an arbitrary function, we define

$$\Lambda(f)(\omega_{ijkl}) = \omega_{f(i)f(j)f(k)f(l)},$$

where $\omega_{ijkl} := 0$ if any two indices are equal. Since this convention makes the defining relations of Λ hold even if some indices coincide, we indeed obtain a homomorphism.

This has important consequences for the S_n-module structure, as the irreducible represention of the category $\mathbf{Fin}^+$ are easily determined (and defined over $\mathbb{Q}$). The irreducible representation theory of $\mathbf{Fin}^+$ is determined by the irreducible representation theory of the "transformation semigroup" (the semigroup of functions from a finite set to itself). Thus from results of [12], we immediately have the following (compare chapter 8 of [11]).

Theorem 5.1. *Let R be an irreducible complex representation of $\mathbf{Fin}^+$. Then precisely one of the following two statements holds for R.*

1 *There exists a nonnegative integer k such that $R(\{1, 2, \dots, n\})$ is $\binom{n-1}{k}$-dimensional, with S_n-character with label $(n-k)1^k$ for $n > k$.*
2 *There exists a partition λ not of the form 1^k such that each S_n-module $R(\{1, 2, \dots, n\})$ is induced from the $S_{|\lambda|} \times S_{n-|\lambda|}$-module in which $S_{n-|\lambda|}$ acts trivially and $S_{|\lambda|}$ acts as the representation λ.*

In particular, we can choose a basis of each $R(S)$ such that all matrix coefficients are rational.

Remark. Note, however, that the transformation semigroup does not have finite representation type, and thus the full representation theory of $\mathbf{Fin}^+$ is wild.

If R is an irreducible representation of $\mathbf{Fin}^+$ with cycle index Z_R, then in the first case we have

$$Z_R = (-1)^k + (e_k - e_{k-1} + e_{k-2} + \dots) \operatorname{Exp},$$

where e_k is the cycle index of the sign representation of S_k, while in the second case we have $Z_R = s_\lambda \operatorname{Exp}$, where s_λ is a Schur function (the cycle index of the irreducible representation indexed by λ).

Corollary 5.2. *If R is a representation of $\mathbf{Fin}^+$ with cycle index Z_R, such that $\dim(R(S)) = O(|S|^l)$ for some integer $l \geq 0$ ("polynomial growth"), then there exists a unique constant C_R such that $\operatorname{Exp}^{-1}(C_R + Z_R)$ is a symmetric function of $\deg_p$ degree at most l.*

Proof. Since $\dim(R(S)) = O(|S|^l)$, the same must be true for the irreducible constituents of R, which must therefore satisfy $k \leq l$ or $|\lambda| \leq l$, as appropriate. The result follows. $\square$

If R is such a representation (or more generally, a graded representation in which each homogeneous component has polynomial growth), we will call

$C_R + Z_R$ the *extended* cycle index of R, and denote it by Z_R^+. Note in particular that $\dim(R(S))$ is polynomial in $|S|$, with constant term C_R.

Corollary 5.3. *The coefficient of t^k in $\operatorname{Exp}^{-1} Z_\Lambda^+$ is a symmetric function of degree at most $3k$.*

Proof. Indeed, the formula for the Poincaré series of Λ implies that the degree k component of $\Lambda(S)$ has dimension $O(n^{3k})$. $\square$

Now, it follows easily from the fact that the S_{n-1}-module Λ'_{n-1} is the restriction of the S_n-module Λ_n that

$$Z_{\Lambda'} = \frac{\partial}{\partial p_1} Z_\Lambda.$$

But this together with the corollary is enough to uniquely determine Z_Λ^+. Indeed, in general, if

$$C_R + Z_R = f \operatorname{Exp}$$

for some symmetric function f of finite degree, then

$$\frac{\partial}{\partial p_1} Z_R = (f + \frac{\partial}{\partial p_1} f) \operatorname{Exp}.$$

If we write f as a polynomial in p_1, we can then solve for its coefficients in order starting with the highest degree term; in other words, the (extended) cycle index of any $\mathbf{Fin}^+$ representation with polynomial growth is uniquely determined by the cycle index of its restriction to point stabilizers.

In our case, we can explicitly solve the corresponding differential equation.

Theorem 5.4. *The extended cycle index of the $\mathbf{Fin}^+$ representation Λ is given by*

$$Z_\Lambda^+ = \frac{-p_1 t + \operatorname{Cosh}^\sim[\operatorname{Arcsinh}^\sim[t^{1/2} p_1]]}{1-t} \operatorname{Exp}[t^{-1/2} \operatorname{Arcsinh}^\sim[t^{1/2} p_1]].$$

In particular, $C_\Lambda = 1/(1-t)$.

Proof. To prove the theorem, we need simply verify that the above expression differentiates to $Z_{\Lambda'}$ and that if we divide by Exp the coefficient of t^k is of bounded degree.

If we divide the above expresion by Exp, we obtain

$$\frac{-p_1 t + \operatorname{Cosh}^\sim[\operatorname{Arcsinh}^\sim[t^{1/2} p_1]]}{1-t} \operatorname{Exp}[t^{-1/2} \operatorname{Arcsinh}^\sim[t^{1/2} p_1] - p_1].$$

Now, if f and g are symmetric functions satisfying the bounded degree condition, then so are $f+g$ and fg; if moreover g has constant term 0 as a series in t, then $f[g]$ has bounded degree coefficients. The second condition follows.

From the identities

$$\frac{\partial}{\partial p_1}\operatorname{Exp}[f] = (\frac{\partial}{\partial p_1} f)\operatorname{Exp}[f],$$
$$\frac{\partial}{\partial p_1}\operatorname{Sinh}^{\sim}[f] = (\frac{\partial}{\partial p_1} f)\operatorname{Cosh}^{\sim}[f],$$
$$\frac{\partial}{\partial p_1}\operatorname{Cosh}^{\sim}[f] = (\frac{\partial}{\partial p_1} f)\operatorname{Sinh}^{\sim}[f],$$

we find, differentiating the defining equation for $\operatorname{Arcsinh}^{\sim}$, that

$$(\frac{\partial}{\partial p_1}\operatorname{Arcsinh}^{\sim})\operatorname{Cosh}^{\sim}[\operatorname{Arcsinh}^{\sim}] = 1$$

and can then immediately verify that Z^+_Λ differentiates as required. □

Remarks. 1. The above formula was guessed via the corresponding formula for the (super) Poincaré series (i.e., setting $p_1 = u$, $p_2 = p_3 = \cdots = 0$):

$$\frac{-ut + \cosh(\operatorname{arcsinh}(u\sqrt{t}))}{1-t}\exp(\operatorname{arcsinh}(u\sqrt{t})/\sqrt{t}).$$

2. Getzler [6] gave the complex analogue, as follows. If we subtract p_1 from the cycle index of $H^*(\overline{M_{0,|S|}}, \mathbb{C})$ we get the unique solution Z of

$$Z = p_1\frac{\partial}{\partial p_1}Z - F[\frac{\partial}{\partial p_1}Z],$$
$$F = p_1\frac{\partial}{\partial p_1}F - Z[\frac{\partial}{\partial p_1}F]$$

where

$$F = \frac{\operatorname{Exp}[(1+t^2)\operatorname{Log}[1+p_1]] - (1+p_1)(1+t^2p_1)}{t^2-t^6} + \frac{h_2}{1+t^2} + \frac{p_1^2-p_2}{2}$$

(essentially the cycle index for $M_{0,|S|}(\mathbb{C})$; note that the formula given in [6] is slightly incorrect, but the correct formula follows from the results of [9]). More precisely, the full cycle index can be expressed as

$$p_1\operatorname{Exp}[C] - F[\operatorname{Exp}[C]-1],$$

where C is as in Theorem 3.6 above. The equations relating Z and F constitute an involutory transformation (the "Legendre transform") integrating the plethystic inverse. With this in mind, we note that the Legendre transform of $Z^+_\lambda - (1-t)^{-1} - p_1$ is

$$\frac{(1+p_1)(\operatorname{Cosh}^{\sim}[t^{1/2}\operatorname{Log}[1+p_1]] - t^{-1/2}\operatorname{Sinh}^{\sim}[t^{1/2}\operatorname{Log}[1+p_1]]) - 1}{1-t},$$

which again presumably has a cohomological interpretation.

This also allows us to prove the remaining cases of Theorem 1.1.

Proof. The point is that if R is any representation of $\mathbf{Fin}^+$ with polynomial growth, then the character of R depends polynomially on the numbers n_i of i-cycles (since this holds for irreducibles). In particular, for Λ we may thus extrapolate to the case with no fixed points. $\square$

If we set $t = 1$ in the formula for the graded character, we obtain the Euler character of M_n. This is straightforward except in the case $n_1 = -1$, when we have a factor $1/(1-t)$ that must be cancelled. We obtain the following result.

Corollary 5.5. *The Euler character χ_E of M_n at the permutation $\pi \in S_n$ is nonzero if and only if one of the following (disjoint) conditions is satisfied. We suppose π has $n_1 + 1$ fixed points and n_l l-cycles for $l \geq 2$.*

1. *π has a fixed point. Then π has order a power of 2 and there exists $k \geq 0$ with $n_1 = n_2 = \ldots n_{2^{k-1}} = 1$, n_{2^k} even. In this case,*

$$\chi_E(\pi) = 2^{k(k-1)/2} \prod_{k \leq j} (1 + o_{2^j}) \prod_{0 \leq i \leq n_{2^j} - 2} (1 + o_{2^j} + 2^j(n_{2^j} - 2 - 2i)).$$

2. *π has no fixed points. Then there exists a nonnegative integer $d > 1$ such that n_d is odd, and every cycle of π has length $2^j d$ for some j. In this case,*

$$\chi_E(\pi) = \Big(\sum_{odd\ k|d} \frac{\mu(k)d}{2k} \Big) \prod_{0 \leq i \leq n_d - 2} d(n_d - 2 - 2i)$$
$$\prod_{1 \leq j} (1 + o_{2^j d}) \prod_{0 \leq i \leq n_{2^j d} - 2} (1 + o_{2^j d} + 2^j d(n_{2^j d} - 2 - 2i)).$$

Proof. If π has a fixed point, then we may simply set $t = 1$ in the formula for the graded character:

$$\prod_{1 \leq l} (\gamma_l(1) + o_l) \prod_{0 \leq i \leq n_l - 2} (\gamma_l(1) + (o_l + l(n_l - 2 - 2i))),$$

where we recall that $\gamma_l(1) = 0$ unless l is a power of 2, in which case $\gamma_l(1) = 1$. Suppose π did not have order a power of 2; then in particular it would have a cycle of length not a power of 2. Let d be the length of the shortest such cycle. Then $o_d = \gamma_d(1) = 0$, and the contribution of the $l = d$ factor of the above product is 0.

Similarly, let k be the smallest integer such that $n_{2^k} \neq 1$. Then $o_{2^k} = 2^k - 1$, and the contribution for $l = 2^k$ is

$$2^k \prod_{0 \leq i \leq n_{2^k} - 2} (2^k + 2^k(n_{2^k} - 2 - 2i)).$$

If n_{2^k} were odd, then the factor for $i = (n_{2^k} - 1)/2$ would make the product 0, and thus n_{2^k} must be even. (In particular, it follows that $n/2^k$ is odd.) The above formula for the Euler character is then straightforward. Since n_{2^k}

is even, it follows that $o_{2j}/2^k$ is odd for all j, and thus none of the remaining factors can vanish.

Now, suppose π has no fixed points. In this case, the contribution for $l = 1$ to the graded character is a factor $1/(1-t)$, and thus rather than avoid all factors that vanish for $t = 1$, we must have exactly one vanishing factor. Suppose d is the length of the shortest cycle of π. Then $\gamma_d(1)+o_d = 0$ (since either d is a power of 2, with $o_d = n_1 = -1$, or d is not a power of 2, and $o_d = 0$), and thus it provides that vanishing factor (and provides more than one unless n_d is odd; in particular n/d must be odd). If there were a cycle of any length not of the form $2^j d$, the shortest such cycle would provide *another* vanishing factor. The above formulae for the Euler character are again straightforward. □

Remarks. 1. For the Euler characteristic itself, the above criterion translates to the statement that $\chi = 0$ unless n is odd (i.e., $n_1 = n-1$ is even), in which case the Euler characteristic is

$$\prod_{0\leq i\leq n-3}(1+(n-3-2i)) = \prod_{0\leq i\leq \lfloor (n-3)/2\rfloor}(1-(n-3-2i)^2),$$

agreeing with the calculations of [3, Thm. 3.2.3] and [5].

2. It should be possible to prove this directly by studying the fixed point set of the action of π on M_n.

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